

Graphene in a strong magnetic field: Massless Dirac particles versus skyrmions

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We discuss models for massless Dirac fermions being subjected to a perpendicular magnetic field in spherical geometry. These models are analogs of Haldane's spherical construction for massful charge carriers. The single-particle states constructed here are easily implemented in existing numerical code for many-body problems in conventional quantum Hall systems. Moreover, the many-body states of fully filled sublevels in the subspace of lowest Landau-level index are skyrmions with respect to the layer spin.

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I. INTRODUCTION

In the recent years, graphene has developed to one of clearly most active directions of work in today's both experimental and theoretical condensed-matter physics.¹⁻³ Compared to conventional two-dimensional electronic systems, the peculiar properties of graphene mainly stem from its linear dispersion near the Fermi energy and the chiral nature of electronic states entangling the momentum and sublattice degree of freedom.^{2,3} Among a plethora of interesting phenomena, the quantum Hall effect occurring at anomalous ("half-integer") filling factors is one of the most spectacular observations in this new type of material.⁴⁻⁶ Other partially related unusual features of graphene include a cyclotron mass being proportional to the square root of the particle density⁴ and a nonequidistant Landau-level spectrum proportional to the square root of the magnetic field.⁴⁻⁹

In quantum Hall physics, the exact numerical treatment of finite many-body systems has always been an important source of theoretical evidence.¹⁰ A particularly convenient model for such numerical simulations was given already 25 years ago by Haldane.¹¹ In this construction, massful electrons move on a spherical surface in a radial monopole magnetic field. This model has the advantage of lacking any system edge (and therefore reducing finite-size effects in numerical results). On the other hand, the mathematical properties of the states in the lowest Landau level are particularly simple, enabling also substantial analytical progress. Both aspects have made the Haldane sphere to a widely used model in theoretical, in particular numerical descriptions of quantum Hall systems; for some representative references see Refs. 12–29. Moreover, most recently this spherical model, which is explicitly constructed for massful electrons in conventional two-dimensional systems, was also applied to the situation of *massless* carriers in graphene.³⁰⁻³⁴ In particular, the authors of Ref. 32 state that "the analogous solution for carriers with linear dispersion ... is not known for the spherical geometry." The purpose of this paper is to introduce such models of massless charge carriers in the spherical geometry. As we shall see below, they are rather straightforwardly developed from the situation of massful particles. A particular feature of these models is many-body ground states having the features of skyrmions known from conventional quantum Hall monolayers. Finally we mention previous works by other authors on Dirac fermions coupled to gauge fields in spherical geometry inspired by

fullerenes,³⁵⁻³⁷ i.e., another allotrope of carbon. Differently from the models to be discussed here, these constructions do not rely on angular-momentum operators and have therefore also different spectra.

This paper is organized as follows. In Sec. II we review the basic properties of the spherical model for massive charge carriers in a perpendicular magnetic field. In Sec. III we briefly recall elementary facts concerning planar graphene in a perpendicular magnetic field before constructing analogous models in spherical geometry. Here we also discuss two-body and many-body states. We close with conclusions in Sec. IV.

II. SPHERICAL MODEL FOR MASSFUL CHARGE CARRIERS

In Haldane's spherical model for massful charge carriers in a perpendicular magnetic field, electrons move on a spherical surface of radius R which is penetrated by a radial monopole magnetic field $B = \hbar c S / e R^2$. Here $e > 0$ is the elementary charge and $2S$ is the integer number of flux quanta hc/e through the surface,¹¹ i.e., the radius is given by $R = \ell \sqrt{S}$, where $\ell = \sqrt{\hbar c / e B}$ is the magnetic length. The single-particle Hamiltonian reads

$$\mathcal{H}_0 = \frac{\vec{\Lambda}^2}{2MR^2} = \frac{1}{2} \omega_c \frac{\vec{\Lambda}^2}{\hbar S}. \quad (1)$$

Here M is an effective mass, $\omega = eB/Mc$ is the cyclotron frequency, and the kinetic angular momentum reads, using again standard notation,

$$\vec{\Lambda} = \vec{r} \times \left(\vec{p} + \frac{e}{c} \vec{A} \right), \quad (2)$$

where the vector potential yields $\nabla \times \vec{A} = B \hat{\Omega}$ and $\hat{\Omega} = \vec{r}/R$. The operator Λ has the elementary properties

$$\vec{\Lambda} \cdot \vec{\Omega} = \vec{\Omega} \cdot \vec{\Lambda} = 0, \quad (3)$$

$$[\Lambda^\alpha, \Lambda^\beta] = i \hbar \varepsilon^{\alpha\beta\gamma} (\Lambda^\gamma - \hbar S \Omega^\gamma). \quad (4)$$

In particular, the components of Λ fail to fulfill proper angular-momentum commutation relations and can therefore not be considered as the generators of rotations. Instead, rotations are generated by the angular-momentum operator

$$\vec{L} = \vec{\Lambda} + \hbar S \vec{\Omega} \quad (5)$$

fulfilling

$$[L^\alpha, L^\beta] = i\hbar \varepsilon^{\alpha\beta\gamma} L^\gamma, \quad (6)$$

$$\vec{L} \cdot \vec{\Omega} = \vec{\Omega} \cdot \vec{L} = \hbar S, \quad (7)$$

$$\vec{L}^2 = \vec{\Lambda}^2 + (\hbar S)^2. \quad (8)$$

Thus, the eigenvalues of \vec{L}^2 are given by $\hbar^2 l(l+1)$ with $l = S+n$ and $n \in \{0, 1, 2, \dots\}$, where $n=0$ corresponds to the lowest Landau level. Note, however, that the spectrum of the Hamiltonian (1) is not equidistant but grows quadratically with n differently from the planar situation.

Let us choose the usual gauge $\vec{A} = (\hbar S / eR) \hat{\phi} \cot \vartheta$, where ϑ and ϕ are the usual polar coordinates and $\hat{\phi}$ is the unit vector in the azimuthal direction.^{11,12} Then eigenstates in the lowest Landau level have a particularly simple structure given by

$$\varphi_m(u, v) = \sqrt{\frac{2S+1}{4\pi\ell^2 S}} \binom{2S}{S+m} u^{S+m} v^{S-m}, \quad (9)$$

where $u = \cos(\vartheta/2)e^{i\phi/2}$, $v = \sin(\vartheta/2)e^{-i\phi/2}$, and $m \in \{-S, \dots, S\}$ is the eigenvalue of L^z / \hbar .

We note that the gauge-invariant angular-momentum operator (5) crucially depends on the magnetic flux and is not identical to the gauge-dependent canonical angular-momentum operator $\vec{L}_{\text{can}} = \vec{r} \times \vec{p}$. Moreover, due to relation (8), one could alternatively define the Hamiltonian

$$\mathcal{H}'_0 = \frac{1}{2} \omega_c \frac{\vec{L}^2}{\hbar S}, \quad (10)$$

which differs from Eq. (1) just by a trivial constant.

III. MASSLESS DIRAC PARTICLES IN SPHERICAL GEOMETRY

For a *planar* graphene sheet in a perpendicular magnetic field, the single-particle states around one of one of the two inequivalent corners of the first Brillouin zone are described by³

$$\mathcal{H}_{(\pm)}^{\text{pl}} = v[(\pm)\pi_x \sigma^x + \pi_y \sigma^y] \quad (11)$$

with $\vec{\pi} = \vec{p} + e\vec{A}/c$ and $v \approx 10^6$ m/s. The Pauli matrices describe the sublattice or pseudospin degree of freedom, and the Zeeman coupling to the physical electron spin has been neglected. The double sign (\pm) (valley index) determines which corner of the Brillouin zone is considered.³⁸ These two cases are, in the absence of a magnetic field, related by time reversal³ implemented by a complex conjugation, i.e., σ^y changes sign while σ^x remains unaltered. Note that this behavior is different from angular-momentum operator describing a proper spin and not a sublattice degree of freedom. In what follows we shall concentrate on the case (+). Defining the usual bosonic operators

$$a = \frac{1}{\sqrt{2}} \frac{\ell}{\hbar} (\pi_x + i\pi_y), \quad a^\dagger = (a)^\dagger \quad (12)$$

fulfilling $[a, a^\dagger] = 1$, the Hamiltonian reads

$$\mathcal{H}_{(+)}^{\text{pl}} = \frac{\hbar v}{\ell} \sqrt{2} (a \sigma^- + a^\dagger \sigma^+), \quad (13)$$

where $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$. The well-known eigenstates³ of the Hamiltonian (13) are given by $|0, \uparrow\rangle$ with energy $\varepsilon_0 = 0$ and, for $n > 0$,

$$|n, \pm\rangle = \frac{1}{\sqrt{2}} (|n, \uparrow\rangle \pm |n-1, \downarrow\rangle) \quad (14)$$

with energy $\varepsilon_n^\pm = \pm (\hbar v / \ell) \sqrt{2n}$. Here n is again the Landau-level index and the arrows are obvious standard notation for the sublattice spin states. In particular, Landau-level index and (sublattice) spin are entangled in these eigenstates, a feature that will be reproduced by the spherical models for graphene in a magnetic field to be discussed now.

A. Spherical models

Let us now discuss models for massless Dirac particles on a sphere penetrated by a radial monopole magnetic field, for which we shall use the same gauge as stated before. Note that the crucial step in constructing the spherical model for massful carriers is to replace the linear momentum of the usual planar Landau problem with an appropriately defined angular momentum. This shall also be our guideline regarding the case of massless Dirac fermions.

1. Single-particle Hamiltonians

Inspired by the expressions (1) and (10) we consider the Hamiltonians

$$\mathcal{H}_{(\pm)} = \frac{v}{R} \vec{\Lambda} \cdot \vec{\tau}_{(\pm)} = \frac{v}{\ell \sqrt{S}} \vec{\Lambda} \cdot \vec{\tau}_{(\pm)} \quad (15)$$

and

$$\mathcal{H}'_{(\pm)} = \frac{v}{\ell \sqrt{S}} \vec{L} \cdot \vec{\tau}_{(\pm)}. \quad (16)$$

Here the operators $\vec{\tau}_{(\pm)}$ describing the sublattice degree of freedom are again just given by the Pauli matrices, $\vec{\tau}_{(+)} = \vec{\sigma}$, while $\vec{\tau}_{(-)}$ are the negatives of their complex conjugates, $\vec{\tau}_{(-)} = -(\vec{\sigma})^*$, in close analogy to the planar case (11). Note that both sets of operators, although not referring to a proper spin, fulfill the usual relations

$$[\tau_{(\pm)}^\alpha, \tau_{(\pm)}^\beta] = 2i\varepsilon^{\alpha\beta\gamma} \tau_{(\pm)}^\gamma. \quad (17)$$

This fact will chiefly facilitate the analysis of the Hamiltonian (16) in terms of elementary angular-momentum theory. The treatment of the Hamiltonian (15), however, is more complicated since the components of $\vec{\Lambda}$ do not fulfill an angular-momentum algebra. Moreover, in the following we shall again concentrate on the case (+). The case (−) can be treated completely analogously and is just related via complex conjugation.

The two Hamiltonians given in Eqs. (15) and (16) differ by the radial contribution

$$\mathcal{H}'_{(+)} - \mathcal{H}_{(+)} = \frac{\hbar v}{\ell} \sqrt{S} \vec{\Omega} \cdot \vec{\sigma}, \quad (18)$$

which is, differently from expressions (1) and (10), not just a trivial constant. In fact, one might argue that $\mathcal{H}_{(+)}$ should be considered to be closer to the planar model of graphene since the operator $\vec{\Lambda}$, in contrast to \vec{L} , does not have a radial component. However, as we shall see shortly, when concentrating on the subspace of lowest Landau-level index $n=0$, the solutions of negative energy are simultaneous eigenstates of $\vec{L} \cdot \vec{\sigma}$ and $\vec{\Omega} \cdot \vec{\sigma}$ (and therefore also $\vec{\Lambda} \cdot \vec{\sigma}$) at any systems size, whereas the solutions of positive energy become such simultaneous eigenstates in the thermodynamic limit $S \rightarrow \infty$. Thus, as far as the subspace of lowest Landau-level index is concerned, the eigenstates of $\mathcal{H}_{(+)}$ and $\mathcal{H}'_{(+)}$ are either identical at any system size or become identical in the thermodynamic limit, where the planar model of graphene is recovered. Let us therefore first concentrate on the latter Hamiltonian.

Introducing a total angular momentum of the usual form $\vec{J} = \vec{L} + \hbar \vec{\sigma} / 2$ one can rewrite the Hamiltonian (16) as

$$\mathcal{H}'_{(+)} = \frac{v}{\ell \hbar \sqrt{S}} \left[\vec{J}^2 - \vec{L}^2 - \left(\frac{\hbar}{2} \vec{\sigma} \right)^2 \right]. \quad (19)$$

Moreover, \vec{J} commutes with the Hamiltonian and admits total angular-momentum quantum numbers $j = l \pm 1/2 = S + n \pm 1/2$. Thus, the spectrum of $\mathcal{H}'_{(+)}$ reads

$$\varepsilon'_{\pm} = \pm \frac{\hbar v}{\ell \sqrt{S}} \left(S + n + \frac{1}{2} \mp \frac{1}{2} \right). \quad (20)$$

Let us again focus on the lowest Landau-level index $n=0$. Using the well-known Clebsch-Gordan coefficients³⁹ for coupling an angular momentum of length S with a spin $1/2$, one can explicitly formulate the $2S+2$ eigenstates in the multiplet $j=l+1/2$,

$$\begin{aligned} \psi_m^+(u, v) &= \sqrt{\frac{S+1/2+m}{2S+1}} \varphi_{m-1/2}(u, v) |\uparrow\rangle \\ &+ \sqrt{\frac{S+1/2-m}{2S+1}} \varphi_{m+1/2}(u, v) |\downarrow\rangle, \end{aligned} \quad (21)$$

where $m \in \{-S-1/2, \dots, S+1/2\}$ is the eigenvalue of J^z / \hbar , and the $2S$ states with $j=l-1/2$ read

$$\begin{aligned} \psi_m^-(u, v) &= -\sqrt{\frac{S+1/2-m}{2S+1}} \varphi_{m-1/2}(u, v) |\uparrow\rangle \\ &+ \sqrt{\frac{S+1/2+m}{2S+1}} \varphi_{m+1/2}(u, v) |\downarrow\rangle \end{aligned} \quad (22)$$

with $m \in \{-S+1/2, \dots, S-1/2\}$. Similarly to the planar model of graphene, the sublattice spin and the conventional orbital motion are entangled with each other.

Denoting $\psi_m^{\pm}(u, v) = \langle \vec{r} | m, \pm \rangle$, the expectation values of $\vec{\Omega} \cdot \vec{\sigma}$ are straightforwardly calculated as

$$\langle m, + | \vec{\Omega} \cdot \vec{\sigma} | m, + \rangle = \frac{2S}{2S+2}, \quad (23)$$

$$\langle m, - | \vec{\Omega} \cdot \vec{\sigma} | m, - \rangle = -1. \quad (24)$$

Thus, the variance of $\vec{\Omega} \cdot \vec{\sigma}$ within the states of lower energy $|m, -\rangle$ is exactly zero, while for the states $|m, +\rangle$ one finds

$$\begin{aligned} &\langle m, + | (\vec{\Omega} \cdot \vec{\sigma})^2 | m, + \rangle - (\langle m, + | \vec{\Omega} \cdot \vec{\sigma} | m, + \rangle)^2 \\ &= 1 - \frac{(2S)^2}{(2S+2)^2} = 4 \frac{2S+1}{(2S+2)^2}. \end{aligned} \quad (25)$$

As a result, the eigenstates $|m, -\rangle$ of the Hamiltonian $\mathcal{H}'_{(+)}$ are for any system size simultaneously also eigenstates of $\mathcal{H}_{(+)}$, whereas the eigenstates $|m, +\rangle$ achieve this property in the thermodynamic limit $S \rightarrow \infty$. In particular the energy ε_- of $\mathcal{H}_{(+)}$ corresponding to ε'_{-} for $n=0$ is just $\varepsilon_- = -\hbar v / \ell \sqrt{S}$. Moreover, replacing, as an approximation, the operator $\vec{\Omega} \cdot \vec{\sigma}$ with ± 1 one finds for the energy spectrum of $\mathcal{H}_{(+)}$

$$\varepsilon_{\pm} \approx \pm \frac{\hbar v}{\ell \sqrt{S}} \left(n + \frac{1}{2} \mp \frac{1}{2} \right). \quad (26)$$

This result should be seen in analogy to the conventional spherical model (1). Here the energies increase quadratically with the Landau-level index n , while the spectrum of the underlying planar model is of course equidistant with energies proportional to n . In the case of graphene, the energies of the planar model are proportional to \sqrt{n} , while in the spherical model $\mathcal{H}_{(+)}$ they increase with $\sqrt{n^2} = n$. As seen above, the approximation underlying Eq. (26) is exact for both branches of the spectrum at $n=0$ and $S \rightarrow \infty$. It is an interesting speculation whether it becomes also exact in the thermodynamic limit if $n \neq 0$. The single-particle states given above were also examined some time ago by Rezayi⁴⁰ in circumstances of quantum Hall skyrmions, an analogy we shall explore in some detail in Sec. III A 3.

2. Two-body states and interaction matrix elements

For the conventional spherical model for massful charge carriers described in Sec. II, a particularly simple form for two-body states of given total angular momentum can be devised.^{11,12} Using these expressions, matrix elements of rotationally invariant interactions are conveniently parameterized in terms of pseudopotentials.¹²

In the present case of massless particles, the pseudospin degree of freedom adds to the complexity of the two-body problem, and we have not found a similarly concise expression for states with good quantum numbers of the total angular momentum. However, what is usually needed in numerical implementations of interaction Hamiltonians are matrix elements between tensor products of single-particle states which are related to pseudopotentials via Clebsch-Gordan coefficients.¹² Now, using the expansions (21) and (22) it is straightforward to express such interaction matrix elements of states of massless particles considered here in terms of those of massful objects given by Eq. (9). Thus, regarding numerical implementations of interaction opera-

tors, it is an easy and straightforward task to adjust an existing code for the conventional spherical model to the single-particle states of massless carriers.

3. Many-body states

We now discuss many-body states of fully filled energetic sublevels (20) of lowest Landau-level index $n=0$. Let $|\Psi^\pm\rangle$ denote the Slater determinant of all $2S+1 \pm 1$ single-particle states given in Eqs. (21) and (22), respectively. Both many-body states are singlets of the total angular momentum. For the particle density one naturally finds

$$n^\pm(\vec{r}) = \langle \Psi^\pm | \sum_i \delta(\vec{r} - \vec{r}_i) | \Psi^\pm \rangle = \frac{2S+1 \pm 1}{4\pi\ell^2 S}, \quad (27)$$

while the pseudospin density is given by

$$\vec{\sigma}^\pm(\vec{r}) = \langle \Psi^\pm | \sum_i \vec{\sigma}_i \delta(\vec{r} - \vec{r}_i) | \Psi^\pm \rangle = \pm \frac{2S}{4\pi\ell^2 S} \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}, \quad (28)$$

in accordance with Eqs. (23) and (24). We see that these pseudospin densities form the typical ‘‘hedgehog’’ structures known from skyrmions.^{10,17} However, differently from skyrmions in conventional quantum Hall monolayers, the physical electron spin as well as the valley spin is polarized here while the sublattice spin is forming a topologically nontrivial structure. In fact, the single-particle wave functions (21) and (22) along with the many-body states $|\Psi^\pm\rangle$ were discussed already in Ref. 40 as models for (electron-spin) skyrmions. Here we have identified formally the same skyrmion states (with respect to the sublattice spin) as ground states of fully filled Landau levels of massless Dirac particles. We stress the fact that the Hilbert spaces spanned by the single-particle states (21) and (22) for massless Dirac particles are different from the Hilbert space spanned by the single-particle wave functions (9) for conventional massful carriers. In several recent publications, however, the Landau levels of graphene were modeled by the conventional wave functions for massive particles.^{30–34}

Finally, in order to evaluate interaction terms within the many-body states $|\Psi^\pm\rangle$ it is useful to consider the pair distribution function

$$g^\pm(|\vec{r}_1 - \vec{r}_2|) = \langle \Psi^\pm | \sum_{i \neq j} \delta(\vec{r}_1 - \vec{r}_i) \delta(\vec{r}_2 - \vec{r}_j) | \Psi^\pm \rangle. \quad (29)$$

Here we obtain

$$g^+(r) = \frac{(2S+2)^2}{(4\pi\ell^2 S)^2} \left[1 - \frac{(2S)^2}{(2S+2)^2} \left(1 - \frac{r^2}{2\ell^2 2S} \right)^{2S-1} - \frac{4S+2}{(2S+2)^2} \left(1 - \frac{r^2}{2\ell^2 2S} \right)^{2S} \right] \quad (30)$$

and

$$g^-(r) = \frac{(2S)^2}{(4\pi\ell^2 S)^2} \left[1 - \left(1 - \frac{r^2}{2\ell^2 2S} \right)^{2S-1} \right]. \quad (31)$$

From these expressions the ground-state energies for arbitrary two-body interactions can be evaluated by integration. In particular, for Coulomb interaction we find

$$E^+ = - \frac{e^2}{\epsilon\ell} \frac{1}{\sqrt{S}} \frac{2^{4S-1}}{\binom{4S}{2S}} \left(S + \frac{4S+2}{4S+1} \right), \quad (32)$$

$$E^- = - \frac{e^2}{\epsilon\ell} \frac{1}{\sqrt{S}} \frac{2^{4S-1}}{\binom{4S}{2S}} S. \quad (33)$$

Here ϵ is the dielectric constant of the host material, and we have as usual assumed that the direct (Hartree) contribution to the ground-state energy is cancelled against a neutralizing background. Result (33) agrees with the one given in Ref. 40, while Eq. (32) differs in detail from statements made there by a contribution which, however, vanishes in the thermodynamic limit. If the states $|\Psi^\pm\rangle$ are interpreted as charged excitations of a conventional quantum Hall ground state at a filling factor of unity, their energies should be compared with conventional ground-state energy given by²⁶

$$E^0 = - \frac{e^2}{\epsilon\ell} \frac{2S+1}{\sqrt{S}} \frac{2^{4S}}{\binom{4S+2}{2S+1}} S. \quad (34)$$

To provide a meaningful comparison,¹⁷ one also has to take into account that the particle number in the excited states $|\Psi^\pm\rangle$ differs from the conventional ground state by ± 1 . Then one finds for the excitation gap in the thermodynamic limit $S \rightarrow \infty$

$$E^\pm - \frac{2S+1 \pm 1}{2S+1} E^0 \rightarrow \frac{e^2}{\epsilon\ell} \frac{1}{4} \sqrt{\frac{\pi}{2}}, \quad (35)$$

in accordance again with Ref. 40, which is exactly the result predicted by field-theoretical considerations.¹⁷ Moreover, in Ref. 40 it was also found numerically that the state $|\Psi^+\rangle$ has a vanishing variance for Coulomb interaction operator in the thermodynamic limit; i.e., this state becomes an eigenstate of the interaction operator in the limit of an infinite system. This surprising result was also reported for other types of long-range interactions.⁴⁰

IV. CONCLUSIONS AND OUTLOOK

We have introduced spherical models for the massless Dirac charge carriers of graphene being subjected to a perpendicular magnetic field. The Hamiltonians (15) and (16) presented here are analogs of Haldane’s spherical construction for massful charge carriers. While the first Hamiltonian (15) is arguably closer to the planar model of graphene, the latter one (16) can be analyzed easily by elementary angular-momentum theory. Both Hamiltonians differ by a nontrivial operator. However, in the subspace of lowest Landau-level

index $n=0$ the eigenstates of the single-particle Hamiltonian (16) become also eigenstates of Eq. (15). The latter result holds for the states of positive energy in the thermodynamic limit, while for the states of negative energy this statement is true at arbitrary system size. In particular, the Hilbert spaces spanned by the single-particle eigenstates of the Hamiltonians introduced here are different from the Hilbert space spanned by the single-particle wave functions for conventional massful carriers. It is a very interesting question for future work, whether recently reported results of numerical studies of quantum Hall physics in graphene using conventional wave functions for massive particles³⁰⁻³⁴ are possibly altered if single-particle states for massless charge carriers are used. Indeed, the single-particle states constructed here

are easily implemented in existing numerical code for many-body problems in conventional quantum Hall systems. Finally, the many-body states of fully filled sublevels in the subspace of lowest Landau-level index are skyrmions with respect to the sublattice spin.

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- ¹K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, *Science* **306**, 666 (2004).
- ²A. K. Geim and K. S. Novoselov, *Nature Mater.* **6**, 183 (2007).
- ³A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, arXiv:0709.1163, *Rev. Mod. Phys.* (to be published)
- ⁴K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, *Nature (London)* **438**, 197 (2005).
- ⁵Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, *Nature (London)* **438**, 201 (2005).
- ⁶K. S. Novoselov, Z. Jiang, Y. Zhang, S. V. Morozov, H. L. Stormer, U. Zeitler, J. C. Maan, G. S. Boebinger, P. Kim, and A. K. Geim, *Science* **315**, 1379 (2007).
- ⁷R. S. Deacon, K. C. Chuang, R. J. Nicholas, K. S. Novoselov, and A. K. Geim, *Phys. Rev. B* **76**, 081406(R) (2007).
- ⁸Z. Jiang, E. A. Henriksen, L. C. Tung, Y.-J. Wang, M. E. Schwartz, M. Y. Han, P. Kim, and H. L. Stormer, *Phys. Rev. Lett.* **98**, 197403 (2007).
- ⁹Z. Jiang, Y. Zhang, H. L. Stormer, and P. Kim, *Phys. Rev. Lett.* **99**, 106802 (2007).
- ¹⁰D. Yoshioka, *The Quantum Hall Effect* (Springer, New York, 2002).
- ¹¹J. M. Parpia and T. L. Rhodes, *Phys. Rev. Lett.* **51**, 805 (1983).
- ¹²G. Fano, F. Ortolani, and E. Colombo, *Phys. Rev. B* **34**, 2670 (1986).
- ¹³F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **54**, 237 (1985).
- ¹⁴F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **60**, 956 (1988).
- ¹⁵E. H. Rezayi and F. D. M. Haldane, *Phys. Rev. Lett.* **61**, 1985 (1988).
- ¹⁶E. H. Rezayi, *Phys. Rev. B* **43**, 5944 (1991).
- ¹⁷K. Moon, H. Mori, K. Yang, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka, and S.-C. Zhang, *Phys. Rev. B* **51**, 5138 (1995).
- ¹⁸X. G. Wu and J. K. Jain, *Phys. Rev. B* **51**, 1752 (1995).
- ¹⁹G. Lazzari, H. Nishioka, E. Vigezzi, and R. A. Broglia, *Phys. Rev. B* **53**, 1064 (1996).
- ²⁰K. Yang, K. Moon, L. Belkhir, H. Mori, S. M. Girvin, A. H. MacDonald, L. Zheng, and D. Yoshioka, *Phys. Rev. B* **54**, 11644 (1996).
- ²¹R. H. Morf, *Phys. Rev. Lett.* **80**, 1505 (1998).
- ²²J. Schliemann and A. H. MacDonald, *Phys. Rev. Lett.* **84**, 4437 (2000).
- ²³J. Schliemann, S. M. Girvin, and A. H. MacDonald, *Phys. Rev. Lett.* **86**, 1849 (2001).
- ²⁴G. Ramon, A. Mann, and E. Cohen, *Phys. Rev. B* **67**, 045323 (2003).
- ²⁵R. H. Morf, N. dAmbrumenil, and S. Das Sarma, *Phys. Rev. B* **66**, 075408 (2002).
- ²⁶J. Schliemann, *Phys. Rev. B* **67**, 035328 (2003).
- ²⁷S. H. Simon, E. H. Rezayi, and M. V. Milovanovic, *Phys. Rev. Lett.* **91**, 046803 (2003).
- ²⁸A. E. Feiguin, E. Rezayi, K. Yang, C. Nayak, and S. Das Sarma, arXiv:0804.4502 (unpublished).
- ²⁹G. Möller, S. H. Simon, and E. H. Rezayi, *Phys. Rev. Lett.* **101**, 176803 (2008).
- ³⁰V. M. Apalkov and T. Chakraborty, *Phys. Rev. Lett.* **97**, 126801 (2006).
- ³¹C. Töke, P. E. Lammert, V. H. Crespi, and J. K. Jain, *Phys. Rev. B* **74**, 235417 (2006).
- ³²C. Töke and J. K. Jain, *Phys. Rev. B* **75**, 245440 (2007).
- ³³C. Töke and J. K. Jain, *Phys. Rev. B* **76**, 081403(R) (2007).
- ³⁴N. Shibata and K. Nomura, *Phys. Rev. B* **77**, 235426 (2008).
- ³⁵V. A. Osipov and E. A. Kochetov, *JETP* **72**, 199 (2000).
- ³⁶D. V. Kolesnikov and V. A. Osipov, *Eur. Phys. J. B* **49**, 465 (2006).
- ³⁷M. Pudlak, R. Pincak, and V. A. Osipov, *Phys. Rev. B* **74**, 235435 (2006).
- ³⁸In order to distinguish this valley index (\pm) from other double signs occurring later on, we always put it into parentheses.
- ³⁹See, e.g., J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, MA, 1994).
- ⁴⁰E. H. Rezayi, *Phys. Rev. B* **56**, R7104 (1997).